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LETTER TO THE EDITOR

Conflict of conservation laws in cyclotron radiation

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Abstract. Cyclotron radiation carries a continuous flux of energy, momentum and angular momentum. Current theory cannot explain how such quantities are derived from the emitting particles (electrons).

For an electron undergoing cyclotron emission in a uniform and constant magnetic field (z direction by convention) the following basic conservation laws must hold as a result of the configurational symmetry of the Hamiltonian.

Energy conservation. The total Hamiltonian of an interacting system of radiation and an electron does not have explicit time dependence. Energy is conserved. We may write

$$\mathrm{d}E_{\mathrm{e}}/\mathrm{d}t = -\mathrm{d}E_{\gamma}/\mathrm{d}t \tag{1}$$

where

$$\mathrm{d}E_{\mathrm{e}}/\mathrm{d}t = \mathrm{d}(\tfrac{1}{2}mr_1^2\omega_{\mathrm{c}}^2)/\mathrm{d}t \tag{2}$$

(r_1 is the radius of the electron orbit, ω_{c} is the cyclotron frequency) and $\mathrm{d}E_{\gamma}/\mathrm{d}t$, the rate of energy loss to radiation, is given in the appendix.

Angular momentum conservation. In the absence of radiation, the gauge-invariant generator of infinitesimal rotation about the z axis is given for the electron as (Johnson and Lippmann 1949)

$$L_z^{\mathrm{e}} = \tfrac{1}{2}m\omega_{\mathrm{c}}(r_1^2 - r_0^2) \tag{3}$$

where $r_0^2 = x_0^2 + y_0^2$ is the radial position of the electron guiding centre. L_z^{e} is conserved because of azimuthal symmetry. In the presence of radiation, the symmetry property remains in the Hamiltonian. The total angular momentum of (electron + radiation) is conserved, i.e.

$$\mathrm{d}L_z^{\mathrm{e}}/\mathrm{d}t = -\mathrm{d}L_z^{\gamma}/\mathrm{d}t. \tag{4}$$

In cyclotron emission, the vector potential for the radiation at the mth harmonic is given in cylindrical polar coordinates as

$$(\boldsymbol{A})_m = [\boldsymbol{\varepsilon}_+ J_{m-1}(k_\perp\rho)\,\mathrm{e}^{\mathrm{i}(m-1)\phi} + \boldsymbol{\varepsilon}_- J_{m+1}(k_\perp\rho)\,\mathrm{e}^{\mathrm{i}(m+1)\phi}]\exp[\mathrm{i}(k_z z - \omega t)]$$

where $J_m(x)$ is the Bessel function, $k_\perp^2 = k_x^2 + k_y^2$, and $\boldsymbol{\varepsilon}_\pm$ are polarisation vectors describing circularly polarised radiation. For such fields we have $L_z^{\gamma} = m$. More importantly, there exists a definite relationship between the energy and angular momentum carried away by the mth *harmonic* radiation (Jackson 1975)

$$(\mathrm{d}L_z^\gamma/\mathrm{d}t)_m = (m/\omega)(\mathrm{d}E_\gamma/\mathrm{d}t)_m$$

and since $m/\omega = 1/\omega_c$ is a constant, the following equation holds between the *total* energy and angular momentum:

$$\mathrm{d}L_z^\gamma/\mathrm{d}t = \omega_c^{-1}\ \mathrm{d}E_\gamma/\mathrm{d}t. \tag{5}$$

Equations (1), (4) and (5) then imply

$$\mathrm{d}L_z^e/\mathrm{d}t = \omega_c^{-1}\ \mathrm{d}E_e/\mathrm{d}t.$$

This, together with (2) and (3), would give

$$\mathrm{d}r_0^2/\mathrm{d}t = 0 \tag{6}$$

as a consequence of energy and angular momentum conservation. The physical meaning is that, during radiation, there is no systematic drift in the radial position of the guiding centre.

Linear momentum conservation. In the absence of radiation, the Hamiltonian for the electron

$$H = (2m)^{-1}[\boldsymbol{p} + (e/c)\boldsymbol{A}]^2$$

(where $A_z = 0$ for the field $H = H_z$) is translationally invariant. Thus the corresponding generators of infinitesimal translation (Lieu 1980, Herold *et al* 1981)

$$P_x^e = m\omega_c y_0, \qquad P_y^e = -m\omega_c x_0, \qquad P_z^e = p_z,$$

are conserved. Moreover, they all have gauge-invariant meanings. P_x^e and P_y^e are respectively proportional to the y and x coordinates of the guiding centre.

During interaction with radiation, the conserved quantity is no longer the electron momentum P_e, but the total electron and radiation field momentum $\boldsymbol{\Pi} = \boldsymbol{P}_e + \boldsymbol{P}_\gamma$. This has been demonstrated explicitly (Avron *et al* 1978). Ignoring Π_z and dropping the suffix γ for radiation, we may write down the components of $\boldsymbol{\Pi}$ as

$$\Pi_x = m\omega_c y_0 + P_x, \qquad \Pi_y = -m\omega_c x_0 + P_y. \tag{7}$$

For the purpose of the present paper it is important to enter cylindrical geometry and construct the following quantity for the electron,

$$r_0^2 = x_0^2 + y_0^2 = (m\omega_c)^{-2}[\Pi^2 - 2(\Pi_x P_x + \Pi_y P_y) + P^2] \tag{8}$$

where $\Pi^2 = \Pi_x^2 + \Pi_y^2$ etc. We now take time derivatives of (7) and (8) to yield

$$\frac{\mathrm{d}y_0}{\mathrm{d}t} = -\frac{1}{m\omega_c}\frac{\mathrm{d}P_x}{\mathrm{d}t}, \qquad \frac{\mathrm{d}x_0}{\mathrm{d}t} = \frac{1}{m\omega_c}\frac{\mathrm{d}P_y}{\mathrm{d}t}, \tag{9a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}r_0^2 = (m\omega_c)^{-2}\left[\frac{\mathrm{d}}{\mathrm{d}t}P^2 - 2\left(\Pi_x\frac{\mathrm{d}P_x}{\mathrm{d}t} + \Pi_y\frac{\mathrm{d}P_y}{\mathrm{d}t}\right)\right]. \tag{9b}$$

Now we interpret (9) physically. $\mathrm{d}P_x/\mathrm{d}t$ and $\mathrm{d}P_y/\mathrm{d}t$ are respectively the rates in which x and y momentum are carried away by the radiation. Owing to azimuthal symmetry, such quantities must be zero (as will be justified in the appendix). However, $(\mathrm{d}P^2/\mathrm{d}t)$ is the rate in which *radial* (ρ) momentum is dissipated. That is, in general, finite.

Thus we have from (9*b*)

$$\mathrm{d}r_0^2/\mathrm{d}t = (m\omega_c)^{-2}\,\mathrm{d}P^2/\mathrm{d}t. \tag{10}$$

Demonstration of the finiteness of $(\mathrm{d}P^2/\mathrm{d}t)$ will also be given in the appendix. But the crucial point is to realise that (10) implies an 'outward drift' of the guiding centre, and is *fundamentally incompatible* with (6).

Within the scope of current knowledge there appears no simple way of resolving the paradox.

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Appendix

The linear momentum terms $\mathrm{d}P_x/\mathrm{d}t$, $\mathrm{d}P_y/\mathrm{d}t$ and $\mathrm{d}P^2/\mathrm{d}t$ of the radiation may be calculated directly from the Maxwell stress tensor (see Jackson 1975, p 239). A simpler way of arriving at the main conclusions in the text is to start from the known formula of the energy radiated in a given harmonic m, per unit solid angle and per unit time (Bekefi 1966)

$$j(\omega, \theta) = (e^2\omega^2/8\pi^2\varepsilon_0 c)[\cot^2\theta\, J_m^2(m\beta\sin\theta) + \beta^2 J_m'^2(m\beta\sin\theta)]$$

where $\beta = v/c$ and v is the (transverse) velocity of the electron, θ is the pitch angle of radiation propagation, and $\omega = m\omega_c$. Multiplication of $j(\omega, \theta)$ by $(1/c)\sin\theta\cos\phi$, $(1/c)\sin\theta\sin\phi$, followed by integration over $\mathrm{d}\Omega = \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi$ and summation over all harmonics, gives respectively the expressions $\mathrm{d}P_x/\mathrm{d}t$ and $\mathrm{d}P_y/\mathrm{d}t$, namely

$$\frac{\mathrm{d}P_x}{\mathrm{d}t} = \frac{e^2}{8\pi^2\varepsilon_0 c^2}\sum_{m=1}^{\infty}\omega^2\int \mathrm{d}\theta\,\mathrm{d}\phi\,[\cos^2\theta\cos\phi\, J_m^2(m\beta\sin\theta) + \beta^2\sin^2\theta\cos\phi\, J_m'^2(m\beta\sin\theta)],$$

$$\frac{\mathrm{d}P_y}{\mathrm{d}t} = \frac{e^2}{8\pi^2\varepsilon_0 c^2}\sum_{m=1}^{\infty}\omega^2\int \mathrm{d}\theta\,\mathrm{d}\phi\,[\cos^2\theta\sin\phi\, J_m^2(m\beta\sin\theta) + \beta^2\sin^2\theta\sin\phi\, J_m'^2(m\beta\sin\theta)].$$

The ϕ integration reveals immediately that $\mathrm{d}P_x/\mathrm{d}t = \mathrm{d}P_y/\mathrm{d}t = 0$. Concerning the finiteness of $\mathrm{d}P^2/\mathrm{d}t$, it is sufficient to show that $\mathrm{d}P/\mathrm{d}t \neq 0$. The latter is obtained from the multiplication of $j(\omega, \theta)$ and $(1/c)\sin\theta$, and summing over solid angle and cyclotron harmonics as before:

$$\frac{\mathrm{d}P}{\mathrm{d}t} = \frac{e^2}{4\pi\varepsilon_0 c^2}\sum_{m=1}^{\infty}\omega^2\int_0^{\pi}\mathrm{d}\theta\,[\cos^2\theta\, J_m^2(m\beta\sin\theta) + \beta^2\sin^2\theta\, J_m'^2(m\beta\sin\theta)].$$

We will not proceed to complete the calculation because it appears sufficiently evident that $\mathrm{d}P/\mathrm{d}t$ is positive definite, as expected in the text.

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